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Minimum-Time Low-Thrust Rendezvous and Transfer Using Epoch Mean Longitude Formulation

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The mathematics of minimum-time rendezvous and transfer using a set of nonsingular orbital elements involving the mean longitude at epoch is presented and shown to represent a basis from which the current time formulations are derived. The state and adjoint differential equations are explicit functions of time in this formulation that involves natural orbital elements that stay constant if no perturbation is applied. The optimal Hamiltonian also is varying in time; however, the function that defines the transversality condition at the end time in minimum-time problems is shown to remain constant during the optimal transfer, effectively replacing the numerical check for the constant Hamiltonian in accepting a converged trajectory as being truly optimal. The mapping matrices for the Lagrange multipliers of this as well as all of the other formulations previously developed also are shown explicitly. A numerical example is presented that uses the shooting method coupled with a quasi-Newtonian scheme to solve the two-point boundary-value problem and generate the numerically integrated trajectory.

Nomenclature

a	= semimajor axis, km
E	= eccentric anomaly
e	= eccentricity
F	= eccentric longitude, $E+\omega+\Omega$
F'	= $1+G=1/\beta$
f	= thrust magnitude, N
$\boldsymbol{f}$	= thrust vector, N
$\hat{\boldsymbol{f}}, \hat{\boldsymbol{g}}, \hat{\boldsymbol{w}}$	= unit vectors along axes of equinoctial frame
G	= $(1-h^2-k^2)^{1/2}$
$\boldsymbol{g}$	= gravitational acceleration vector, km/s^2
i	= orbital inclination
K	= $1+p^2+q^2$
L	= true longitude, $\theta^*+\omega+\Omega$
M	= mean anomaly
M_0	= mean anomaly at epoch
m	= spacecraft mass, kg
n	= orbital mean motion, $\mu^{1/2}a^{-3/2}$, rad/s
r	= radial distance, km
$\dot{\boldsymbol{r}}$	= velocity vector, km/s
$\ddot{\boldsymbol{r}}$	= acceleration vector, km/s^2
s_F, c_F	= $\sin F$, $\cos F$, etc.
$\hat{\boldsymbol{u}}$	= unit vector in the direction of $\boldsymbol{f}$
v	= $\lvert\dot{\boldsymbol{r}}\rvert$, velocity vector magnitude, km/s
β	= $1/(1+G)$
θ^*	= true anomaly
λ	= mean longitude, $M+\omega+\Omega$
λ_0	= mean longitude at epoch, $M_0+\omega+\Omega$
μ	= Earth gravitational constant, km^3/s^2
Ω	= right ascension of ascending node
ω	= argument of perigee

Introduction

THE use of the nonsingular equinoctial orbital elements in solving the problem of optimal orbital transfer using low-thrust acceleration was initiated by Edelbaum et al.[1] The full set of the state and adjoint differential equations for the case in which the current mean longitude is chosen as the fast element appeared in Refs. 2 and 3 by extending the analysis of Cefola[4] and Cefola et al.[5] and Edelbaum et al.[1] The low-thrust transfer and rendezvous problem using continuous constant acceleration with the current mean longitude as the fast variable[6] is solved using the epoch mean longitude as the sixth state variable instead, this element being a natural orbital element that stays constant in the absence of any perturbation acceleration. This epoch formulation is fundamental in the sense that the current time formulations are derived directly from it. Its development is of considerable academic as well as practical engineering interest inasmuch as it constitutes a basis for the generation of nonsingular trajectory optimization software that are calculus-of-variations based and that use the equinoctial elements as coordinates. The transversality condition at the unknown final time needed in the numerical solution of the two-point boundary-value problem for minimum-time transfers is more complicated than for the case in which the current mean longitude is selected as the fast variable. Nevertheless, the seven-parameter search is established to satisfy the boundary conditions and generate converged optimal transfer and rendezvous trajectories, duplicating previously generated results for mutual validation.

Equations of Motion for Epoch Mean Longitude Formulation

This section provides a general description of the equations of motion for the fundamental set of equinoctial elements by first showing how the partial derivatives of these elements, with respect to the position and velocity vectors, are derived from the knowledge of the Poisson brackets of the elements, and thereby constructing the required set of the variation-of-parameters equations. This general analysis is shown here for completeness, with many of the important expressions for the components of the position and velocity vectors in the direct equinoctial reference frame defined later, with reference to the original contributions in Refs. 4, 5, and 7. Let Table 1 define the correspondence between the fundamental classical and equinoctial element sets.

Now, letting $z=(a\ \ h\ \ k\ \ p\ \ q\ \ \lambda_0)^T$ represent the state vector at time t and $z=f(\boldsymbol{r},\dot{\boldsymbol{r}})$,

$$\dot{z}=\frac{\partial z}{\partial \boldsymbol{r}}\dot{\boldsymbol{r}}+\frac{\partial z}{\partial \dot{\boldsymbol{r}}}\ddot{\boldsymbol{r}} \tag{1}$$

where

$$\ddot{\boldsymbol{r}}=(\boldsymbol{f}/m)+\boldsymbol{g}=(\boldsymbol{f}/m)-(\mu/r^3)\boldsymbol{r} \tag{2}$$

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Table 1 Classical and equinoctial element transformations

$h = e\sin(\omega+\Omega)$	$e = (h^2+k^2)^{1/2}$
$k = e\cos(\omega+\Omega)$	$i = 2\tan^{-1}(p^2+q^2)^{1/2}$
$p = \tan(i/2)\sin\Omega$	$\Omega = \tan^{-1}(p/q)$
$q = \tan(i/2)\cos\Omega$	$\omega = \tan^{-1}(h/k) - \tan^{-1}(p/q)$
$\lambda_0 = M_0+\omega+\Omega$	$M_0 = \lambda_0 - \tan^{-1}(h/k)$

The partial derivatives of the equinoctial elements with respect to $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ are related to the partials of $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ with respect to the elements z through the Poisson brackets (a_α, a_β) of the elements by

$$\frac{\partial a_\alpha}{\partial \boldsymbol{r}} = \sum_{\beta=1}^{6}(a_\alpha, a_\beta)\frac{\partial \dot{\boldsymbol{r}}}{\partial a_\beta} \tag{3}$$

$$\frac{\partial a_\alpha}{\partial \dot{\boldsymbol{r}}} = -\sum_{\beta=1}^{6}(a_\alpha, a_\beta)\frac{\partial \boldsymbol{r}}{\partial a_\beta} \tag{4}$$

The Poisson brackets P are related[8] to the Lagrange brackets L by $P = -L^{-1}$ and $P^T = L^{-1}$ with

$$L = \left(\frac{\partial \boldsymbol{r}}{\partial z}\right)^T\frac{\partial \dot{\boldsymbol{r}}}{\partial z} - \left(\frac{\partial \dot{\boldsymbol{r}}}{\partial z}\right)^T\frac{\partial \boldsymbol{r}}{\partial z} \tag{5}$$

$$P = \frac{\partial z}{\partial \boldsymbol{r}}\left(\frac{\partial z}{\partial \dot{\boldsymbol{r}}}\right)^T - \frac{\partial z}{\partial \dot{\boldsymbol{r}}}\left(\frac{\partial z}{\partial \boldsymbol{r}}\right)^T \tag{6}$$

with the Lagrange brackets given by $[a_\alpha, a_\beta] = (\partial \boldsymbol{r}^T/\partial a_\alpha)(\partial \dot{\boldsymbol{r}}/\partial a_\beta) - (\partial \boldsymbol{r}^T/\partial a_\beta)(\partial \dot{\boldsymbol{r}}/\partial a_\alpha)$ and the Poisson brackets given by $(a_\alpha, a_\beta) = (\partial a_\alpha/\partial \boldsymbol{r})(\partial a_\beta/\partial \dot{\boldsymbol{r}})^T - (\partial a_\alpha/\partial \dot{\boldsymbol{r}})(\partial a_\beta/\partial \boldsymbol{r})^T$. The various matrices of interest are given in terms of $\boldsymbol{r} = (x, y, z)$ and $\dot{\boldsymbol{r}} = (\dot{x}, \dot{y}, \dot{z})$, such that, for example,

$$[a,a] = \left(\frac{\partial x}{\partial a}\frac{\partial \dot{x}}{\partial a} - \frac{\partial x}{\partial a}\frac{\partial \dot{x}}{\partial a}\right) + \left(\frac{\partial y}{\partial a}\frac{\partial \dot{y}}{\partial a} - \frac{\partial y}{\partial a}\frac{\partial \dot{y}}{\partial a}\right)$$

$$+\left(\frac{\partial z}{\partial a}\frac{\partial \dot{z}}{\partial a} - \frac{\partial z}{\partial a}\frac{\partial \dot{z}}{\partial a}\right) = 0$$

$$(a,a) = \left(\frac{\partial a}{\partial x}\frac{\partial a}{\partial \dot{x}} - \frac{\partial a}{\partial x}\frac{\partial a}{\partial \dot{x}}\right) + \left(\frac{\partial a}{\partial y}\frac{\partial a}{\partial \dot{y}} - \frac{\partial a}{\partial y}\frac{\partial a}{\partial \dot{y}}\right)$$

$$+\left(\frac{\partial a}{\partial z}\frac{\partial a}{\partial \dot{z}} - \frac{\partial a}{\partial z}\frac{\partial a}{\partial \dot{z}}\right) = 0$$

The various brackets also can be written as the sum of three Jacobian determinants[8,9]

$$[a_\alpha, a_\beta] = \frac{\partial(x,\dot{x})}{\partial(a_\alpha, a_\beta)} + \frac{\partial(y,\dot{y})}{\partial(a_\alpha, a_\beta)} + \frac{\partial(z,\dot{z})}{\partial(a_\alpha, a_\beta)} \tag{7}$$

$$(a_\alpha, a_\beta) = \frac{\partial(a_\alpha, a_\beta)}{\partial(x,\dot{x})} + \frac{\partial(a_\alpha, a_\beta)}{\partial(y,\dot{y})} + \frac{\partial(a_\alpha, a_\beta)}{\partial(z,\dot{z})} \tag{8}$$

where, for example,

$$\frac{\partial(x,\dot{x})}{\partial(a_\alpha, a_\beta)} = \begin{vmatrix} \dfrac{\partial x}{\partial a_\alpha} & \dfrac{\partial x}{\partial a_\beta} \\ \dfrac{\partial \dot{x}}{\partial a_\alpha} & \dfrac{\partial \dot{x}}{\partial a_\beta} \end{vmatrix} = \frac{\partial x}{\partial a_\alpha}\frac{\partial \dot{x}}{\partial a_\beta} - \frac{\partial x}{\partial a_\beta}\frac{\partial \dot{x}}{\partial a_\alpha} \tag{9}$$

The matrix of the Poisson brackets P of the elements a, h, k, p, q, λ_0 can be obtained in a more direct manner[7] by a simple transformation applied to the matrix of the Poisson brackets of the classical elements $a, e, i, \Omega, \omega, M_0$. This mapping is shown in great detail[2,10] as

$$P = \frac{1}{na^2}\begin{bmatrix} 0 & 0 & 0 & -2a & 0 & 0 \\ & 0 & -G & \dfrac{hG}{F'} & \dfrac{-kpK}{2G} & \dfrac{-kqK}{2G} \\ & & 0 & \dfrac{kG}{F'} & \dfrac{hpK}{2G} & \dfrac{hqK}{2G} \\ & & & 0 & \dfrac{-pK}{2G} & \dfrac{-qK}{2G} \\ & & & & 0 & \dfrac{-K^2}{4G} \\ -\mathrm{Sym} & & & & & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (a,a) & (a,h) & (a,k) & (a,\lambda_0) & (a,p) & (a,q) \\ (h,a) & (h,h) & (h,k) & (h,\lambda_0) & (h,p) & (h,q) \\ (k,a) & (k,h) & (k,k) & (k,\lambda_0) & (k,p) & (k,q) \\ (\lambda_0,a) & (\lambda_0,h) & (\lambda_0,k) & (\lambda_0,\lambda_0) & (\lambda_0,p) & (\lambda_0,q) \\ (p,a) & (p,h) & (p,k) & (p,\lambda_0) & (p,p) & (p,q) \\ (q,a) & (q,h) & (q,k) & (q,\lambda_0) & (q,p) & (q,q) \end{bmatrix} \tag{10}$$

Now, using the property

$$\left[\partial\begin{pmatrix}\boldsymbol{r}\\ \dot{\boldsymbol{r}}\end{pmatrix}\Big/\partial z\right]\left[\partial z\Big/\partial\begin{pmatrix}\boldsymbol{r}\\ \dot{\boldsymbol{r}}\end{pmatrix}\right] = I_{(6\times 6)}$$

or, in expanded form,

$$\begin{pmatrix}\dfrac{\partial \boldsymbol{r}}{\partial z}\\ \dfrac{\partial \dot{\boldsymbol{r}}}{\partial z}\end{pmatrix}\begin{pmatrix}\dfrac{\partial z}{\partial \boldsymbol{r}} & \dfrac{\partial z}{\partial \dot{\boldsymbol{r}}}\end{pmatrix} = \begin{bmatrix}\dfrac{\partial \boldsymbol{r}}{\partial z}\dfrac{\partial z}{\partial \boldsymbol{r}} & \dfrac{\partial \boldsymbol{r}}{\partial z}\dfrac{\partial z}{\partial \dot{\boldsymbol{r}}} \\ \dfrac{\partial \dot{\boldsymbol{r}}}{\partial z}\dfrac{\partial z}{\partial \boldsymbol{r}} & \dfrac{\partial \dot{\boldsymbol{r}}}{\partial z}\dfrac{\partial z}{\partial \dot{\boldsymbol{r}}}\end{bmatrix} = \begin{pmatrix} I_{(3\times 3)} & 0_{(3\times 3)} \\ 0_{(3\times 3)} & I_{(3\times 3)}\end{pmatrix} \tag{11}$$

and postmultiplying matrix P in Eq. (6) by $(\partial \dot{\boldsymbol{r}}/\partial z)^T$ and $(\partial \boldsymbol{r}/\partial z)^T$, respectively, will yield $P(\partial \dot{\boldsymbol{r}}/\partial z)^T = \partial z/\partial \boldsymbol{r}$ and $P(\partial \boldsymbol{r}/\partial z)^T = -(\partial z/\partial \dot{\boldsymbol{r}})$ or, in other words, the identities in Eqs. (3) and (4). This is true because

$$\left(\frac{\partial z}{\partial \boldsymbol{r}}\right)^T\left(\frac{\partial \dot{\boldsymbol{r}}}{\partial z}\right)^T = 0_{(3\times 3)}, \qquad \left(\frac{\partial z}{\partial \dot{\boldsymbol{r}}}\right)^T\left(\frac{\partial \boldsymbol{r}}{\partial z}\right)^T = 0_{(3\times 3)}$$

Going back to Eq. (1), writing one component at a time, and making use of Eq. (2),

$$\dot{z} = \frac{\partial z}{\partial \boldsymbol{r}}\dot{\boldsymbol{r}} + \frac{\partial z}{\partial \dot{\boldsymbol{r}}}\ddot{\boldsymbol{r}}$$

$$= \sum_\beta (a_\alpha, a_\beta)\frac{\partial \dot{\boldsymbol{r}}}{\partial a_\beta}\dot{\boldsymbol{r}} - \sum_\beta (a_\alpha, a_\beta)\frac{\partial \boldsymbol{r}}{\partial a_\beta}\ddot{\boldsymbol{r}}$$

$$= \sum_\beta (a_\alpha, a_\beta)\frac{\partial \dot{\boldsymbol{r}}}{\partial a_\beta}\dot{\boldsymbol{r}} + \sum_\beta (a_\alpha, a_\beta)\frac{\partial \boldsymbol{r}}{\partial a_\beta}\left(\frac{\mu}{r^3}\right)\boldsymbol{r} + \frac{\partial z}{\partial \dot{\boldsymbol{r}}}\frac{\boldsymbol{f}}{m}$$

$$= \sum_\beta (a_\alpha, a_\beta)\left(\frac{\partial \dot{\boldsymbol{r}}}{\partial a_\beta}\dot{\boldsymbol{r}} + \frac{\mu}{r^3}\frac{\partial \boldsymbol{r}}{\partial a_\beta}\boldsymbol{r}\right) + \frac{\partial z}{\partial \dot{\boldsymbol{r}}}\frac{\boldsymbol{f}}{m} \tag{12}$$

The term in brackets can be written as

$$\frac{1}{2}\frac{\partial(\dot{\boldsymbol{r}}^2)}{\partial z} + \frac{\mu}{r^3}\frac{1}{2}\frac{\partial(\boldsymbol{r}^2)}{\partial z} = \frac{1}{2}\left(\frac{\partial v^2}{\partial z} + \frac{\mu}{r^3}\frac{\partial r^2}{\partial z}\right) \tag{13}$$

because $|\dot{\boldsymbol{r}}| = v$, and using the energy equation $\frac{1}{2}v^2 - (\mu/r) = C$, which yields

$$\frac{1}{2}\frac{\partial v^2}{\partial z} = -\frac{\mu}{r^2}\frac{\partial r}{\partial z}$$

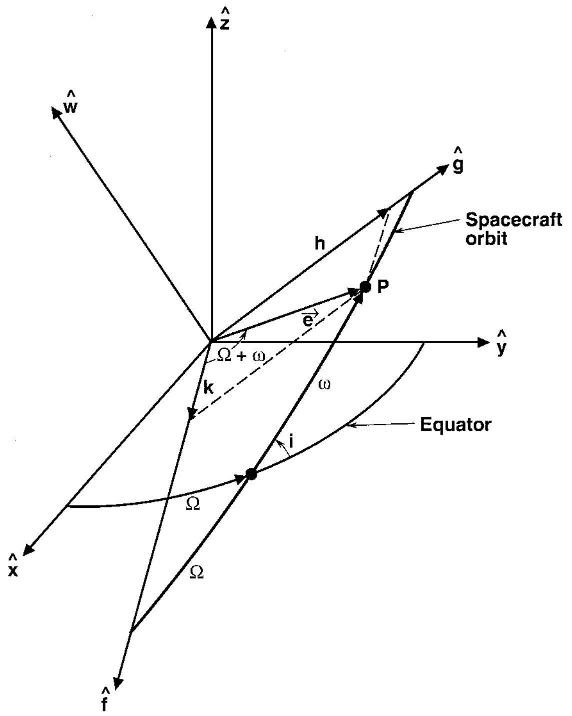

Fig. 1 Direct equinoctial reference frame.

and in view of $\partial r^2/\partial z = 2r\partial r/\partial z$, the bracket in Eq. (13) cancels out such that

$$\dot{z} = \frac{\partial z}{\partial \dot{\boldsymbol{r}}}\frac{f}{m}$$

and in vector form, the variation of parameters equations

$$\dot{z} = \frac{\partial z}{\partial \dot{\boldsymbol{r}}}\frac{f}{m}\hat{u} = \frac{\partial z}{\partial \dot{\boldsymbol{r}}}f_t\hat{u} \tag{14}$$

From Eq. (4), and the known Poisson brackets in Eq. (10), the $\dot{z}$ differential equations above can be obtained. If both $\hat{u}$ and $\partial z/\partial\dot{\boldsymbol{r}}$ are expressed in the direct equinoctial orbital frame with unit vectors $\hat{\boldsymbol{f}}$, $\hat{\boldsymbol{g}}$, $\hat{\boldsymbol{w}}$ shown in Fig. 1, such that $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{g}}$ are contained in the instantaneous orbital plane with the direction of $\hat{\boldsymbol{f}}$ obtained through a clockwise rotation of an angle Ω from the direction of the ascending node, then $\boldsymbol{r}$ and $\dot{\boldsymbol{r}}$ are such that[4,5,7] $\boldsymbol{r} = X_1\hat{\boldsymbol{f}} + Y_1\hat{\boldsymbol{g}}$, and $\dot{\boldsymbol{r}} = \dot{X}_1\hat{\boldsymbol{f}} + \dot{Y}_1\hat{\boldsymbol{g}}$, where the components X_1, Y_1, $\dot{X}_1$, $\dot{Y}_1$ are written in terms of the equinoctial elements and F, and also shown on p. 804 of Ref. 11.

The following position vector partials are used in generating the equations of motion in Eq. (14):

$$\frac{\partial \boldsymbol{r}}{\partial a} = \frac{\partial X_1}{\partial a}\hat{\boldsymbol{f}} + \frac{\partial Y_1}{\partial a}\hat{\boldsymbol{g}} = \left(\frac{X_1}{a} - \frac{3}{2}\frac{t}{a}\dot{X}_1\right)\hat{\boldsymbol{f}} + \left(\frac{Y_1}{a} - \frac{3}{2}\frac{t}{a}\dot{Y}_1\right)\hat{\boldsymbol{g}}$$

$$\frac{\partial \boldsymbol{r}}{\partial h} = \frac{\partial X_1}{\partial h}\hat{\boldsymbol{f}} + \frac{\partial Y_1}{\partial h}\hat{\boldsymbol{g}}, \qquad \frac{\partial \boldsymbol{r}}{\partial k} = \frac{\partial X_1}{\partial k}\hat{\boldsymbol{f}} + \frac{\partial Y_1}{\partial k}\hat{\boldsymbol{g}}$$

$$\frac{\partial \boldsymbol{r}}{\partial \lambda_0} = \frac{\partial \boldsymbol{r}}{\partial F}\frac{\partial F}{\partial \lambda}\frac{\partial \lambda}{\partial \lambda_0} = \frac{\dot{\boldsymbol{r}}}{n}$$

because $\partial\boldsymbol{r}/\partial\lambda = (\dot{X}_1/n)\hat{\boldsymbol{f}} + (\dot{Y}_1/n)\hat{\boldsymbol{g}} = \dot{\boldsymbol{r}}/n$. Finally,

$$\frac{\partial \boldsymbol{r}}{\partial p} = X_1\frac{\partial \hat{\boldsymbol{f}}}{\partial p} + Y_1\frac{\partial \hat{\boldsymbol{g}}}{\partial p} = \frac{2}{K}[q(Y_1\hat{\boldsymbol{f}} - X_1\hat{\boldsymbol{g}}) - X_1\hat{\boldsymbol{w}}]$$

$$\frac{\partial \boldsymbol{r}}{\partial q} = X_1\frac{\partial \hat{\boldsymbol{f}}}{\partial q} + Y_1\frac{\partial \hat{\boldsymbol{g}}}{\partial q} = \frac{2}{K}[p(X_1\hat{\boldsymbol{g}} - Y_1\hat{\boldsymbol{f}}) - Y_1\hat{\boldsymbol{w}}]$$

These expressions make use of the following partial derivatives of F with respect to the elements λ_0, a, h, and k:

$$\frac{\partial F}{\partial \lambda_0} = \frac{a}{r}, \qquad \frac{\partial F}{\partial a} = -\frac{3}{2}\frac{nt}{r}, \qquad \frac{\partial F}{\partial h} = -\frac{a}{r}c_F, \qquad \frac{\partial F}{\partial k} = \frac{a}{r}s_F$$

They are derived easily from $\lambda = \lambda_0 + nt = F - ks_F + hc_F$, which yields

$$\mathrm{d}F = \left[\mathrm{d}\lambda_0 + n\,\mathrm{d}t - \tfrac{3}{2}(nt/a)\,\mathrm{d}a + s_F\,\mathrm{d}k - c_F\,\mathrm{d}h\right](a/r)$$

Also, the partial $\partial\boldsymbol{r}/\partial\lambda_0$ can be obtained directly from

$$\frac{\partial \boldsymbol{r}}{\partial \lambda_0} = \frac{\partial \boldsymbol{r}}{\partial F}\frac{\partial F}{\partial \lambda_0} = \left(\frac{\partial X_1}{\partial F}\hat{\boldsymbol{f}} + \frac{\partial Y_1}{\partial F}\hat{\boldsymbol{g}}\right)\frac{a}{r} = \left(\frac{r}{a}\frac{\dot{X}_1}{n}\hat{\boldsymbol{f}} + \frac{r}{a}\frac{\dot{Y}_1}{n}\hat{\boldsymbol{g}}\right)\frac{a}{r} = \frac{\dot{\boldsymbol{r}}}{n}$$

The partials $\partial\boldsymbol{r}/\partial p$ and $\partial\boldsymbol{r}/\partial q$ are obtained by observing that X_1 and Y_1 are not functions of p and q, but that the unit vectors $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{g}}$ are such functions because $\hat{\boldsymbol{f}}$, $\hat{\boldsymbol{g}}$, and $\hat{\boldsymbol{w}}$ are given in the inertial $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ frame by

$$\hat{\boldsymbol{f}} = \frac{1}{1 + p^2 + q^2}\begin{pmatrix} 1 - p^2 + q^2 \\ 2pq \\ -2p \end{pmatrix}$$

$$\hat{\boldsymbol{g}} = \frac{1}{1 + p^2 + q^2}\begin{pmatrix} 2pq \\ 1 + p^2 - q^2 \\ 2q \end{pmatrix}$$

$$\hat{\boldsymbol{w}} = \frac{1}{1 + p^2 + q^2}\begin{pmatrix} 2p \\ -2q \\ 1 - p^2 - q^2 \end{pmatrix}$$

Therefore,

$$\frac{\partial \boldsymbol{r}}{\partial p} = \frac{2X_1}{(1 + p^2 + q^2)^2}\begin{pmatrix} -2p - 2pq^2 \\ 2q - q(1 + p^2 - q^2) \\ -2q^2 - (1 - p^2 - q^2) \end{pmatrix} + \frac{2Y_1}{(1 + p^2 + q^2)^2}\begin{pmatrix} q(1 - p^2 + q^2) \\ 2pq^2 \\ -2pq \end{pmatrix}$$

which yields the form written above. Similar manipulations yield the expression for $\partial\boldsymbol{r}/\partial q$. Also, the partials $\partial X_1/\partial h$, $\partial X_1/\partial k$, $\partial Y_1/\partial h$, and $\partial Y_1/\partial k$ are generated by using

$$\frac{\partial F}{\partial h} = -\frac{a}{r}c_F, \qquad \frac{\partial F}{\partial k} = \frac{a}{r}s_F$$

and

$$\frac{\partial \beta}{\partial h} = \frac{h\beta^3}{1 - \beta}, \qquad \frac{\partial \beta}{\partial k} = \frac{k\beta^3}{1 - \beta}$$

They are shown explicitly in Refs. 4 and 5 and also on p. 804 of Ref. 11.

Carrying out the algebra results in the generation of the partials of the five slowly varying elements a, h, k, p, and q with respect to the velocity vector, namely, $\partial a/\partial\dot{\boldsymbol{r}}$, $\partial h/\partial\dot{\boldsymbol{r}}$, $\partial k/\partial\dot{\boldsymbol{r}}$, $\partial p/\partial\dot{\boldsymbol{r}}$, and $\partial q/\partial\dot{\boldsymbol{r}}$ shown, e.g., in Eqs. (7–11) of Ref. 11, and by $\partial\lambda_0/\partial\dot{\boldsymbol{r}}$ given below:

$$\begin{aligned}\frac{\partial \lambda_0}{\partial \dot{\boldsymbol{r}}} &= n^{-1}a^{-2}\left[-2X_1 + 3\dot{X}_1 t + G\left(h\beta\frac{\partial X_1}{\partial h} + k\beta\frac{\partial X_1}{\partial k}\right)\right]\hat{\boldsymbol{f}} \\ &+ n^{-1}a^{-2}\left[-2Y_1 + 3\dot{Y}_1 t + G\left(h\beta\frac{\partial Y_1}{\partial h} + k\beta\frac{\partial Y_1}{\partial k}\right)\right]\hat{\boldsymbol{g}} \\ &+ n^{-1}a^{-2}G^{-1}(qY_1 - pX_1)\hat{\boldsymbol{w}} = M_{61}^0\hat{\boldsymbol{f}} + M_{62}^0\hat{\boldsymbol{g}} + M_{63}^0\hat{\boldsymbol{w}}\end{aligned} \tag{15}$$

The magnitude of $\boldsymbol{r}$ is simply obtained from the orbital equation $r = a(1 - kc_F - hs_F)$. The system of differential equations to integrate then is given by

$$\dot{a} = \left(\frac{\partial a}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t, \qquad \dot{h} = \left(\frac{\partial h}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t$$

$$\dot{k} = \left(\frac{\partial k}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t, \qquad \dot{p} = \left(\frac{\partial p}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t$$

$$\dot{q} = \left(\frac{\partial q}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t, \qquad \dot{\lambda}_0 = \left(\frac{\partial \lambda_0}{\partial \dot{\boldsymbol{r}}}\right)^T \cdot \hat{\boldsymbol{u}} f_t$$

where n is osculating too because it is dependent on the semimajor axis. Because we are integrating the epoch mean longitude λ_0 as the sixth element and because X_1, Y_1, $\dot{X}_1$, $\dot{Y}_1$, r as well as the partials $\partial X_1/\partial h$ through $\partial Y_1/\partial k$ are given in terms of the eccentric longitude F, it is necessary at each integration step to solve Kepler's equation after evaluating the current mean longitude λ from $\lambda = \lambda_0 + nt$:

$$\lambda = F - ks_F + hc_F \tag{16}$$

for F, through a Newton–Raphson iteration scheme. The error tolerance is set to 10^{-15}.

Variational Hamiltonian

This section provides an explicit form of the differential equations for the adjoint variables, which in turn show how these variables remain truly constant in the absence of perturbations for this particular set. The assumptions needed in generating these equations also are discussed, leading to the form of the partial derivatives of the eccentric longitude accessory variable with respect to the four elements upon which it is dependent, namely, the elements λ_0, h, k, and a, the semimajor axis.

The Hamiltonian of the system is given by

$$H = \boldsymbol{\lambda}_z^T \dot{\boldsymbol{z}} = \boldsymbol{\lambda}_z^T M(\boldsymbol{z}, F) f_t \hat{\boldsymbol{u}} \tag{17}$$

where the seventh differential equation for the mass flow rate has been neglected because of the small expenditure of propellant during the transfer for a spacecraft with a low-thrust propulsion system. The Euler–Lagrange equations are given by

$$\dot{\boldsymbol{\lambda}}_z = -\frac{\partial H}{\partial \boldsymbol{z}} = -\boldsymbol{\lambda}_z^T \frac{\partial M}{\partial \boldsymbol{z}} f_t \hat{\boldsymbol{u}} \tag{18}$$

The optimal thrust direction $\hat{\boldsymbol{u}}$ is chosen such that it is at all times parallel to $\boldsymbol{\lambda}_z^T M(\boldsymbol{z}, F)$ in order to maximize the Hamiltonian. The matrix M for the set $(a, h, k, p, q, \lambda_0)$ is as given by Eqs. (7–11) of Ref. 11 and by Eq. (15) of this paper. The partials $\partial M/\partial \boldsymbol{z}$ are shown in the Appendix. If we let u_i represent the components of $\hat{\boldsymbol{u}}$ in the equinoctial frame, then with $u_1 = u_f$, $u_2 = u_g$, and $u_3 = u_w$ and using the summation notation:

$$H = \left[\lambda_a(M_{1i}u_i) + \lambda_h(M_{2i}u_i) + \lambda_k(M_{3i}u_i) + \lambda_p(M_{4i}u_i) + \lambda_q(M_{5i}u_i) + \lambda_{\lambda_0}\left(M_{6i}^0 u_i\right)\right] f_t \tag{19}$$

The corresponding Lagrange equations are simply given by the partial derivatives of the above Hamiltonian with respect to the six elements of interest:

$$\dot{\lambda}_a = -\left(\lambda_a \quad \lambda_h \quad \lambda_k \quad \lambda_p \quad \lambda_q \quad \lambda_{\lambda_0}\right) \frac{\partial M}{\partial a} f_t \hat{\boldsymbol{u}} = -\frac{\partial H}{\partial a} = \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial a}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial a}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial a}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial a}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial a}u_i\right) - \lambda_{\lambda_0}\left(\frac{\partial M_{6i}^0}{\partial a}u_i\right)\right] f_t \tag{20}$$

$$\dot{\lambda}_h = -\frac{\partial H}{\partial h} = \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial h}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial h}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial h}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial h}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial h}u_i\right) - \lambda_{\lambda_0}\left(\frac{\partial M_{6i}^0}{\partial h}u_i\right)\right] f_t \tag{21}$$

$$\dot{\lambda}_k = -\frac{\partial H}{\partial k} = \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial k}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial k}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial k}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial k}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial k}u_i\right) - \lambda_{\lambda_0}\left(\frac{\partial M_{6i}^0}{\partial k}u_i\right)\right] f_t \tag{22}$$

$$\dot{\lambda}_p = -\frac{\partial H}{\partial p} = \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial p}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial p}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial p}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial p}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial p}u_i\right) - \lambda_{\lambda_0}\left(\frac{\partial M_{6i}^0}{\partial p}u_i\right)\right] f_t \tag{23}$$

$$\dot{\lambda}_q = -\frac{\partial H}{\partial q} = \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial q}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial q}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial q}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial q}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial q}u_i\right) - \lambda_{\lambda_0}\left(\frac{\partial M_{6i}^0}{\partial q}u_i\right)\right] f_t \tag{24}$$

$$\dot{\lambda}_{\lambda_0} = -\frac{\partial H}{\partial \lambda_0} = \left[-\lambda_a\left(\frac{\partial M_{1i}}{\partial \lambda_0}u_i\right) - \lambda_h\left(\frac{\partial M_{2i}}{\partial \lambda_0}u_i\right) - \lambda_k\left(\frac{\partial M_{3i}}{\partial \lambda_0}u_i\right) - \lambda_p\left(\frac{\partial M_{4i}}{\partial \lambda_0}u_i\right) - \lambda_q\left(\frac{\partial M_{5i}}{\partial \lambda_0}u_i\right) - \lambda_{\lambda_0}\left(\frac{\partial M_{6i}^0}{\partial \lambda_0}u_i\right)\right] f_t \tag{25}$$

All partials of the matrix M with respect to the elements appearing in the above differential equations are shown in the Appendix. The following partials are used to generate the $\partial M/\partial \boldsymbol{z}$ partials of the Appendix. In this formulation, F is a function of λ_0, h, k, and a, because, from Kepler's equation $\lambda_0 + nt = F - ks_F + hc_F$, we have the equations in Table 2.

Table 2 Partials of *F* and *r* with respect to orbital elements

$\dfrac{\partial F}{\partial \lambda_0} = \dfrac{a}{r}$	$\dfrac{\partial r}{\partial F} = a(ks_F - hc_F)$
$\dfrac{\partial F}{\partial a} = -\dfrac{3}{2}\dfrac{nt}{r}$	$\dfrac{\partial r}{\partial a} = \dfrac{r}{a} - \dfrac{3}{2}\dfrac{nat}{r}(ks_F - hc_F)$
$\dfrac{\partial F}{\partial h} = -\dfrac{a}{r}c_F$	$\dfrac{\partial r}{\partial h} = \dfrac{a^2}{r}(h - s_F)$
$\dfrac{\partial F}{\partial k} = \dfrac{a}{r}s_F$	$\dfrac{\partial r}{\partial k} = \dfrac{a^2}{r}(k - c_F)$

The partials of the first five rows of the M matrix with respect to h, k, p, and q are identical to those in Ref. 11, and they are not repeated here. Because F is dependent on a, all of the partials with respect to a involve the explicit appearance of terms in t. In minimum-fuel problems, all of the state and adjoint variables remain constant during the coasting arc when the present λ_0 formulation is used. The switching function, which determines the locations where the thrust must be either turned on or turned off, becomes a function of the current anomaly on the conic orbit corresponding to the coast arc. The numerical interpolation needed to find the zeros of the switching function that correspond to the beginning of the powered arcs thus is carried out easily with the present λ_0 formulation. Also, in the context of averaging, the averaged rate $\dot{\tilde{\lambda}}_a$ involves only one Gaussian–Legendre quadrature as opposed to two quadratures for the λ formulation of Ref. 11.

Canonical Transformations

One of the main purposes in deriving the various formulations corresponding to different orbital element sets is to mutually validate the mathematics of all of the corresponding differential equations, and the underlying assumptions used in their derivations. These formulations thus are coded and run to generate duplicate transfers and the various state and adjoint variables' evolutions during the given transfer plotted for comparison. Canonical transformations are of great use in mathematically relating the various adjoint variables for further verification purposes and also for selecting the initial conditions for some or all of the adjoint variables for a given set once the solution corresponding to another set is known. This allows for open-loop trajectory runs without the need to solve the two-point boundary problem repeatedly in order to validate a given formulation. To this end, the mappings between all the equinoctial sets analyzed thus far are shown explicitly in this section beginning with the classical and equinoctial pair using the current mean anomaly and current mean longitude as the sixth state variable, respectively. The canonicity condition for this particular transformation is given by

$$\begin{aligned}\lambda_a\,\mathrm{d}a + \lambda_h\,\mathrm{d}h + \lambda_k\,\mathrm{d}k + \lambda_p\,\mathrm{d}p + \lambda_q\,\mathrm{d}q + \lambda_\lambda\,\mathrm{d}\lambda - H^\lambda\,\mathrm{d}t \\ = \lambda_a\,\mathrm{d}a + \lambda_e\,\mathrm{d}e + \lambda_i\,\mathrm{d}i + \lambda_\Omega\,\mathrm{d}\Omega + \lambda_\omega\,\mathrm{d}\omega \\ + \lambda_M\,\mathrm{d}M - H^M\,\mathrm{d}t\end{aligned} \tag{26}$$

Examples of canonical transformations can be found in Ref. 12. From the inverse transformation given in Table 1, and $M = \lambda - \tan^{-1}(h/k)$, we have, respectively,

$$\mathrm{d}e = \frac{h}{e}\,\mathrm{d}h + \frac{k}{e}\,\mathrm{d}k, \qquad \mathrm{d}i = \frac{2c_{i/2}^2}{\tan(i/2)}p\,\mathrm{d}p + \frac{2c_{i/2}^2}{\tan(i/2)}q\,\mathrm{d}q$$

$$\mathrm{d}\Omega = c_\Omega^2\frac{(q\,\mathrm{d}p - p\,\mathrm{d}q)}{q^2}, \qquad \mathrm{d}\omega = \frac{(k\,\mathrm{d}h - h\,\mathrm{d}k)}{e^2} - \frac{(q\,\mathrm{d}p - p\,\mathrm{d}q)}{\tan^2(i/2)}$$

$$\mathrm{d}M = \mathrm{d}\lambda - \frac{(k\,\mathrm{d}h - h\,\mathrm{d}k)}{e^2}$$

Inserting these identities in Eq. (26) yields

$$\lambda_a = \lambda_a \tag{27}$$

$$\lambda_h = \lambda_e s_{\omega+\Omega} + (\lambda_\omega/e)c_{\omega+\Omega} - (\lambda_M/e)c_{\omega+\Omega} \tag{28}$$

$$\lambda_k = \lambda_e c_{\omega+\Omega} - (\lambda_\omega/e)s_{\omega+\Omega} + (\lambda_M/e)s_{\omega+\Omega} \tag{29}$$

$$\lambda_p = 2\lambda_i s_\Omega c_{i/2}^2 + \lambda_\Omega\frac{c_\Omega}{\tan(i/2)} - \lambda_\omega\frac{c_\Omega}{\tan(i/2)} \tag{30}$$

$$\lambda_q = 2\lambda_i c_\Omega c_{i/2}^2 - \lambda_\Omega\frac{s_\Omega}{\tan(i/2)} + \lambda_\omega\frac{s_\Omega}{\tan(i/2)} \tag{31}$$

$$\lambda_\lambda = \lambda_M \tag{32}$$

$$H^M = H^\lambda \tag{33}$$

Here H^M stands for the Hamiltonian of the classical system and H^λ for the equinoctial system. Now if we consider the transformation between the set (a, h, k, p, q, λ) and the set $(a, h, k, p, q, \lambda_0)$ and using

$$\mathrm{d}\lambda = \mathrm{d}\lambda_0 + t\,\mathrm{d}n + n\,\mathrm{d}t \tag{34}$$

because $\lambda = \lambda_0 + nt$ and with $\mathrm{d}n = -\frac{3}{2}(n/a)\,\mathrm{d}a$, the canonicity condition requires

$$\lambda_a^\lambda\,\mathrm{d}a + \lambda_\lambda^\lambda\,\mathrm{d}\lambda - H^\lambda\,\mathrm{d}t = \lambda_a^{\lambda_0}\,\mathrm{d}a + \lambda_{\lambda_0}^{\lambda_0}\,\mathrm{d}\lambda_0 - H^{\lambda_0}\,\mathrm{d}t \tag{35}$$

where H^{λ_0} is given by Eq. (19). The superscript λ_0 is used in the right-hand side of the above relation to indicate that it is the one corresponding to the $(a, h, k, p, q, \lambda_0)$ set. This yields

$$\lambda_\lambda^\lambda = \lambda_{\lambda_0}^{\lambda_0} \tag{36}$$

$$H^{\lambda_0} = H^\lambda - n\lambda_\lambda^\lambda = H^\lambda - n\lambda_{\lambda_0}^{\lambda_0} \tag{37}$$

$$\lambda_a^\lambda = \lambda_a^{\lambda_0} + \tfrac{3}{2}(nt/a)\lambda_\lambda^\lambda = \lambda_a^{\lambda_0} + \tfrac{3}{2}(nt/a)\lambda_{\lambda_0}^{\lambda_0} \tag{38}$$

It is also true that $\lambda_{\lambda_0}^{\lambda_0} = \lambda_\lambda^\lambda = \lambda_M = \lambda_{M_0}$ such that Eqs. (28–31) are still valid with λ_M identically replaced by λ_{M_0} in the first two expressions. Now using the superscript F and F_0 for the (a, h, k, p, q, F) and (a, h, k, p, q, F_0) formulations of Refs. 13 and 14, respectively, we have from Ref. 13,

$$\begin{aligned}&\lambda_a^F = \lambda_a^\lambda, \qquad \lambda_h^F = \lambda_h^\lambda + \lambda_\lambda^\lambda c_F, \qquad \lambda_k^F = \lambda_k^\lambda - \lambda_\lambda^\lambda s_F \\ &\lambda_p^F = \lambda_p^\lambda, \qquad \lambda_q^F = \lambda_q^\lambda, \qquad \lambda_F^F = (r/a)\lambda_\lambda^\lambda, \qquad H^F = H^\lambda\end{aligned} \tag{39}$$

and from Ref. 14,

$$\begin{aligned}&\lambda_a^{F_0} = \lambda_a^F - \tfrac{3}{2}(nt/a)(a/r)\lambda_F^F, \qquad \lambda_h^{F_0} = \lambda_h^F - (a/r)c_F\lambda_F^F \\ &\lambda_k^{F_0} = \lambda_k^F + (a/r)s_F\lambda_F^F, \qquad \lambda_p^{F_0} = \lambda_p^F, \qquad \lambda_q^{F_0} = \lambda_q^F \\ &\lambda_{F_0}^{F_0} = (a/r)(r_0/a_0)\lambda_F^F, \qquad H^{F_0} = H^F - n(a/r)\lambda_F^F\end{aligned} \tag{40}$$

Furthermore, from Ref. 15 and the present paper using the $(a, h, k, p, q, \lambda_0)$ formulation, we have

$$\begin{aligned}&\lambda_a^{\lambda_0} = \lambda_a^\lambda - \tfrac{3}{2}(nt/a)\lambda_\lambda^\lambda, \qquad \lambda_h^{\lambda_0} = \lambda_h^\lambda, \qquad \lambda_k^{\lambda_0} = \lambda_k^\lambda \\ &\lambda_p^{\lambda_0} = \lambda_p^\lambda, \qquad \lambda_q^{\lambda_0} = \lambda_q^\lambda, \qquad \lambda_{\lambda_0}^{\lambda_0} = \lambda_\lambda^\lambda, \qquad H^{\lambda_0} = H^\lambda - n\lambda_\lambda^\lambda\end{aligned} \tag{41}$$

The expressions in Eqs. (39) and (41) combine to yield

$$\begin{aligned}&\lambda_a^F = \lambda_a^{\lambda_0} + \tfrac{3}{2}(nt/a)\lambda_{\lambda_0}^{\lambda_0}, \qquad \lambda_h^F = \lambda_h^{\lambda_0} + \lambda_{\lambda_0}^{\lambda_0}c_F \\ &\lambda_k^F = \lambda_k^{\lambda_0} - \lambda_{\lambda_0}^{\lambda_0}s_F, \qquad \lambda_p^F = \lambda_p^{\lambda_0}, \qquad \lambda_q^F = \lambda_q^{\lambda_0} \\ &\lambda_F^F = (r/a)\lambda_{\lambda_0}^{\lambda_0}, \qquad H^F = H^{\lambda_0} + n\lambda_{\lambda_0}^{\lambda_0}\end{aligned} \tag{42}$$

which then are combined with Eqs. (40) to yield

$$\begin{aligned}&\lambda_a^{F_0} = \lambda_a^{\lambda_0}, \qquad \lambda_h^{F_0} = \lambda_h^{\lambda_0}, \qquad \lambda_k^{F_0} = \lambda_k^{\lambda_0}, \qquad \lambda_p^{F_0} = \lambda_p^{\lambda_0} \\ &\lambda_q^{F_0} = \lambda_q^{\lambda_0}, \qquad \lambda_{F_0}^{F_0} = (r_0/a_0)\lambda_{\lambda_0}^{\lambda_0}, \qquad H^{F_0} = H^{\lambda_0}\end{aligned} \tag{43}$$

In the matrix form, the following mappings are obtained:

$$\begin{pmatrix} \lambda_a^{\lambda} \\ \lambda_h^{\lambda} \\ \lambda_k^{\lambda} \\ \lambda_p^{\lambda} \\ \lambda_q^{\lambda} \\ \lambda_{\lambda}^{\lambda} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_{\omega+\Omega} & 0 & 0 & \dfrac{c_{\omega+\Omega}}{e} & \dfrac{-c_{\omega+\Omega}}{e} \\ 0 & c_{\omega+\Omega} & 0 & 0 & \dfrac{-s_{\omega+\Omega}}{e} & \dfrac{s_{\omega+\Omega}}{e} \\ 0 & 0 & 2s_{\Omega}c_{i/2}^2 & \dfrac{c_{\Omega}}{\tan(i/2)} & \dfrac{-c_{\Omega}}{\tan(i/2)} & 0 \\ 0 & 0 & 2c_{\Omega}c_{i/2}^2 & \dfrac{-s_{\Omega}}{\tan(i/2)} & \dfrac{s_{\Omega}}{\tan(i/2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_a \\ \lambda_e \\ \lambda_i \\ \lambda_{\Omega} \\ \lambda_{\omega} \\ \lambda_M \end{pmatrix} \tag{44}$$

with $H^{\lambda} = H_M$.

Equations (41) and (44) combine to yield

$$\begin{pmatrix} \lambda_a^{\lambda_0} \\ \lambda_h^{\lambda_0} \\ \lambda_k^{\lambda_0} \\ \lambda_p^{\lambda_0} \\ \lambda_q^{\lambda_0} \\ \lambda_{\lambda_0}^{\lambda_0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\dfrac{3}{2}\dfrac{nt}{a} \\ 0 & s_{\omega+\Omega} & 0 & 0 & \dfrac{c_{\omega+\Omega}}{e} & \dfrac{-c_{\omega+\Omega}}{e} \\ 0 & c_{\omega+\Omega} & 0 & 0 & \dfrac{-s_{\omega+\Omega}}{e} & \dfrac{s_{\omega+\Omega}}{e} \\ 0 & 0 & 2s_{\Omega}c_{i/2}^2 & \dfrac{c_{\Omega}}{\tan(i/2)} & \dfrac{-c_{\Omega}}{\tan(i/2)} & 0 \\ 0 & 0 & 2c_{\Omega}c_{i/2}^2 & \dfrac{-s_{\Omega}}{\tan(i/2)} & \dfrac{s_{\Omega}}{\tan(i/2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_a \\ \lambda_e \\ \lambda_i \\ \lambda_{\Omega} \\ \lambda_{\omega} \\ \lambda_M \end{pmatrix}$$

with $H^{\lambda_0} = H^{\lambda} - n\lambda_{\lambda}^{\lambda} = H^M - n\lambda_M$.

Equations (39) and (44) combine to yield

$$\begin{pmatrix} \lambda_a^{F} \\ \lambda_h^{F} \\ \lambda_k^{F} \\ \lambda_p^{F} \\ \lambda_q^{F} \\ \lambda_{F}^{F} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & s_{\omega+\Omega} & 0 & 0 & \dfrac{c_{\omega+\Omega}}{e} & \dfrac{-c_{\omega+\Omega}}{e} + c_F \\ 0 & c_{\omega+\Omega} & 0 & 0 & \dfrac{-s_{\omega+\Omega}}{e} & \dfrac{s_{\omega+\Omega}}{e} - s_F \\ 0 & 0 & 2s_{\Omega}c_{i/2}^2 & \dfrac{c_{\Omega}}{\tan(i/2)} & \dfrac{-c_{\Omega}}{\tan(i/2)} & 0 \\ 0 & 0 & 2c_{\Omega}c_{i/2}^2 & \dfrac{-s_{\Omega}}{\tan(i/2)} & \dfrac{s_{\Omega}}{\tan(i/2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{r}{a} \end{pmatrix} \begin{pmatrix} \lambda_a \\ \lambda_e \\ \lambda_i \\ \lambda_{\Omega} \\ \lambda_{\omega} \\ \lambda_M \end{pmatrix}$$

with $H^F = H^{\lambda} = H^M$.

Equations (39), (40), and (44) combine to yield

$$\begin{pmatrix} \lambda_a^{F_0} \\ \lambda_h^{F_0} \\ \lambda_k^{F_0} \\ \lambda_p^{F_0} \\ \lambda_q^{F_0} \\ \lambda_{F_0}^{F_0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\dfrac{3}{2}\dfrac{nt}{a} \\ 0 & s_{\omega+\Omega} & 0 & 0 & \dfrac{c_{\omega+\Omega}}{e} & \dfrac{-c_{\omega+\Omega}}{e} \\ 0 & c_{\omega+\Omega} & 0 & 0 & \dfrac{-s_{\omega+\Omega}}{e} & \dfrac{s_{\omega+\Omega}}{e} \\ 0 & 0 & 2s_{\Omega}c_{i/2}^2 & \dfrac{c_{\Omega}}{\tan(i/2)} & \dfrac{-c_{\Omega}}{\tan(i/2)} & 0 \\ 0 & 0 & 2c_{\Omega}c_{i/2}^2 & \dfrac{-s_{\Omega}}{\tan(i/2)} & \dfrac{s_{\Omega}}{\tan(i/2)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{r_0}{a_0} \end{pmatrix} \begin{pmatrix} \lambda_a \\ \lambda_e \\ \lambda_i \\ \lambda_{\Omega} \\ \lambda_{\omega} \\ \lambda_M \end{pmatrix}$$

with $H^{F_0} = H^F - n(a/r)\lambda_F^F = H^F - n\lambda_\lambda^\lambda = H^F - n\lambda_M = H^\lambda - n\lambda_M = H^{\lambda_0}$ and $H^{F_0} = H^M - n\lambda_M$ with $r_0/a_0 = 1 - k_0 c_{F_0} - h_0 s_{F_0}$. The preceding mapping matrices also can be written in terms of the equinoctial variables and associated quantities by observing that $(p^2+q^2)^{1/2} = (K-1)^{1/2}$, $p^2+q^2 = \tan^2(i/2)$, $s_\Omega = p/(p^2+q^2)^{1/2}$, $c_\Omega = q/(p^2+q^2)^{1/2}$, $c_{i/2}^2 = 1/(1+p^2+q^2)$, such that we have the identities in Table 3.

These transformations lead to

$$\begin{pmatrix} \lambda_a^{\lambda_0} \\ \lambda_h^{\lambda_0} \\ \lambda_k^{\lambda_0} \\ \lambda_p^{\lambda_0} \\ \lambda_q^{\lambda_0} \\ \lambda_{\lambda_0}^{\lambda_0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{3}{2}\frac{nt}{a} \\ 0 & \frac{h\beta}{(2\beta-1)^{\frac{1}{2}}} & 0 & 0 & \frac{k\beta^2}{2\beta-1} & \frac{-k\beta^2}{2\beta-1} \\ 0 & \frac{k\beta}{(2\beta-1)^{\frac{1}{2}}} & 0 & 0 & \frac{-h\beta^2}{2\beta-1} & \frac{h\beta^2}{2\beta-1} \\ 0 & 0 & \frac{2p}{K(K-1)^{\frac{1}{2}}} & \frac{q}{K-1} & \frac{-q}{K-1} & 0 \\ 0 & 0 & \frac{2q}{K(K-1)^{\frac{1}{2}}} & \frac{-p}{K-1} & \frac{p}{K-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_a \\ \lambda_e \\ \lambda_i \\ \lambda_\Omega \\ \lambda_\omega \\ \lambda_M \end{pmatrix} \tag{45}$$

$$\begin{pmatrix} \lambda_a^{F_0} \\ \lambda_h^{F_0} \\ \lambda_k^{F_0} \\ \lambda_p^{F_0} \\ \lambda_q^{F_0} \\ \lambda_{F_0}^{F_0} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{3}{2}\frac{nt}{a} \\ 0 & \frac{h\beta}{(2\beta-1)^{\frac{1}{2}}} & 0 & 0 & \frac{k\beta^2}{2\beta-1} & \frac{-k\beta^2}{2\beta-1} \\ 0 & \frac{k\beta}{(2\beta-1)^{\frac{1}{2}}} & 0 & 0 & \frac{-h\beta^2}{2\beta-1} & \frac{h\beta^2}{2\beta-1} \\ 0 & 0 & \frac{2p}{K(K-1)^{\frac{1}{2}}} & \frac{q}{K-1} & \frac{-q}{K-1} & 0 \\ 0 & 0 & \frac{2q}{K(K-1)^{\frac{1}{2}}} & \frac{-p}{K-1} & \frac{p}{K-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 - k_0 c_{F_0} - h_0 s_{F_0} \end{pmatrix} \begin{pmatrix} \lambda_a \\ \lambda_e \\ \lambda_i \\ \lambda_\Omega \\ \lambda_\omega \\ \lambda_M \end{pmatrix} \tag{46}$$

The last element of the above 6×6 matrix is equal to r_0/a_0 with a_0 fixed and r_0 osculating because F_0 is itself osculating. Both h_0 and k_0 also are fixed and for an initial circular orbit $e_0 = 0$ such that h_0 and k_0 are both equal to 0 and therefore $r_0/a_0 = 1$ will not osculate but will remain constant. Furthermore,

$$\begin{pmatrix} \lambda_a^{F} \\ \lambda_h^{F} \\ \lambda_k^{F} \\ \lambda_p^{F} \\ \lambda_q^{F} \\ \lambda_{F}^{F} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{h\beta}{(2\beta-1)^{\frac{1}{2}}} & 0 & 0 & \frac{k\beta^2}{2\beta-1} & \frac{-k\beta^2}{2\beta-1} + c_F \\ 0 & \frac{k\beta}{(2\beta-1)^{\frac{1}{2}}} & 0 & 0 & \frac{-h\beta^2}{2\beta-1} & \frac{h\beta^2}{2\beta-1} - s_F \\ 0 & 0 & \frac{2p}{K(K-1)^{\frac{1}{2}}} & \frac{q}{K-1} & \frac{-q}{K-1} & 0 \\ 0 & 0 & \frac{2q}{K(K-1)^{\frac{1}{2}}} & \frac{-p}{K-1} & \frac{p}{K-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 - kc_F - hs_F \end{pmatrix} \begin{pmatrix} \lambda_a \\ \lambda_e \\ \lambda_i \\ \lambda_\Omega \\ \lambda_\omega \\ \lambda_M \end{pmatrix} \tag{47}$$

Here, $1 - kc_F - hs_F = r/a$. Finally,

$$\begin{pmatrix} \lambda_a^{\lambda} \\ \lambda_h^{\lambda} \\ \lambda_k^{\lambda} \\ \lambda_p^{\lambda} \\ \lambda_q^{\lambda} \\ \lambda_{\lambda}^{\lambda} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{h\beta}{(2\beta-1)^{\frac{1}{2}}} & 0 & 0 & \frac{k\beta^2}{2\beta-1} & \frac{-k\beta^2}{2\beta-1} \\ 0 & \frac{k\beta}{(2\beta-1)^{\frac{1}{2}}} & 0 & 0 & \frac{-h\beta^2}{2\beta-1} & \frac{h\beta^2}{2\beta-1} \\ 0 & 0 & \frac{2p}{K(K-1)^{\frac{1}{2}}} & \frac{q}{K-1} & \frac{-q}{K-1} & 0 \\ 0 & 0 & \frac{2q}{K(K-1)^{\frac{1}{2}}} & \frac{-p}{K-1} & \frac{p}{K-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_a \\ \lambda_e \\ \lambda_i \\ \lambda_\Omega \\ \lambda_\omega \\ \lambda_M \end{pmatrix} \tag{48}$$

Table 3 Identities for the transformation of the mapping matrices

$s_{\omega+\Omega} = \dfrac{h}{e} = \dfrac{h\beta}{(2\beta-1)^{1/2}}$	$2s_\Omega c_{i/2}^2 = \dfrac{2p}{K(K-1)^{1/2}}$
$c_{\omega+\Omega} = \dfrac{k}{e} = \dfrac{k\beta}{(2\beta-1)^{1/2}}$	$2c_\Omega c_{i/2}^2 = \dfrac{2q}{K(K-1)^{1/2}}$
$\dfrac{c_{\omega+\Omega}}{e} = \dfrac{k\beta^2}{2\beta-1}$	$\dfrac{s_\Omega}{\tan(i/2)} = \dfrac{p}{K-1}$
$\dfrac{s_{\omega+\Omega}}{e} = \dfrac{h\beta^2}{2\beta-1}$	$\dfrac{c_\Omega}{\tan(i/2)} = \dfrac{q}{K-1}$

The transformation matrix between the multipliers of the (a, h, k, p, q, L) and (a, h, k, p, q, λ) sets have been derived in Refs. 16 and 17 such that

$$\begin{pmatrix} \lambda_a^\lambda \\ \lambda_h^\lambda \\ \lambda_k^\lambda \\ \lambda_p^\lambda \\ \lambda_q^\lambda \\ \lambda_\lambda^\lambda \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \left(\dfrac{\partial L}{\partial h}\right)_{\text{tot}} \\ 0 & 0 & 1 & 0 & 0 & \left(\dfrac{\partial L}{\partial k}\right)_{\text{tot}} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{a^2 G}{r^2} \end{pmatrix} \begin{pmatrix} \lambda_a^L \\ \lambda_h^L \\ \lambda_k^L \\ \lambda_p^L \\ \lambda_q^L \\ \lambda_L^L \end{pmatrix}$$

where $(\partial L/\partial h)_{\text{tot}}$ and $(\partial L/\partial k)_{\text{tot}}$ are given either in terms of the true longitude L or the eccentric longitude F by

$$\left(\frac{\partial L}{\partial h}\right)_{\text{tot}} = \frac{(a^2/r^2)G\left[(1-h^2\beta)s_L - hk\beta c_L\right] - s_L - 2(a/r)h}{hc_L - ks_L}$$

$$= \frac{(-a^2/r^2)(s_F - h)G^2 + Y_1/a + 2h}{G(ks_F - hc_F)}$$

$$\left(\frac{\partial L}{\partial k}\right)_{\text{tot}} = \frac{(a^2/r^2)G\left[(1-k^2\beta)c_L - hk\beta s_L\right] - c_L - 2(a/r)k}{hc_L - ks_L}$$

$$= \frac{(-a^2/r^2)(c_F - k)G^2 + X_1/a + 2k}{G(ks_F - hc_F)}$$

The inverse transformation is given by

$$\begin{pmatrix} \lambda_a^L \\ \lambda_h^L \\ \lambda_k^L \\ \lambda_p^L \\ \lambda_q^L \\ \lambda_L^L \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dfrac{-r^2}{a^2G}\left(\dfrac{\partial L}{\partial h}\right)_{\text{tot}} \\ 0 & 0 & 1 & 0 & 0 & \dfrac{-r^2}{a^2G}\left(\dfrac{\partial L}{\partial k}\right)_{\text{tot}} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{r^2}{a^2 G} \end{pmatrix} \begin{pmatrix} \lambda_a^\lambda \\ \lambda_h^\lambda \\ \lambda_k^\lambda \\ \lambda_p^\lambda \\ \lambda_q^\lambda \\ \lambda_\lambda^\lambda \end{pmatrix} \tag{49}$$

The mapping between $(\lambda_a^L, \lambda_h^L, \lambda_k^L, \lambda_p^L, \lambda_q^L, \lambda_L^L)$ and $(\lambda_a, \lambda_e, \lambda_i, \lambda_\Omega, \lambda_\omega, \lambda_M)$ is obtained from Eqs. (48) and (49).

Boundary Conditions for Minimum-Time Rendezvous and Example of a Free-Free Minimum-Time Transfer

In problems of minimum-time orbital rendezvous, rendezvous time is minimized by maximizing the performance index:

$$J = \int_{t_0}^{t_f} L\,\mathrm{d}t = -\int_{t_0}^{t_f} \mathrm{d}t = -(t_f - t_0)$$

with $L = -1$. The two-point boundary-value problem consists of guessing the initial values of the Lagrange multipliers $(\lambda_a^{\lambda_0})_0$, $(\lambda_h^{\lambda_0})_0$, $(\lambda_k^{\lambda_0})_0$, $(\lambda_p^{\lambda_0})_0$, $(\lambda_q^{\lambda_0})_0$, $(\lambda_{\lambda 0}^{\lambda_0})_0$ and the rendezvous time t_f and with given initial parameters a_0, h_0, k_0, p_0, q_0, $(\lambda_0)_0$ and integrating the equations of motion and Eqs. (20–25) using the optimal thrust direction $\hat{\boldsymbol{u}} = (\boldsymbol{\lambda}_z^T M)^T / |\boldsymbol{\lambda}_z^T M|$, such that the final state given by a_f, h_f, k_f, p_f, q_f, λ_f, and a particular transversality condition are matched. The matching of $\lambda_f = (\lambda_0)_f + n_f t_f$ acts as a constraint because λ_f is a function of the state variables λ_0 and a. Following Ref. 18 and letting ν represent a constant multiplier, the transversality condition at the unknown final time t_f is given by

$$\left(\frac{\partial \Phi}{\partial t} + L + \boldsymbol{\lambda}_z^T \dot{\boldsymbol{z}}\right)_{t_f} = 0 \tag{50}$$

where $\Phi = \nu[(\lambda_0)_f + n_f t_f - \lambda_f] = \nu\psi$. We also must require that

$$\left(\lambda_{\lambda_0}^{\lambda_0}\right)_f = \left[\nu \frac{\partial \psi}{\partial \lambda_0}\right]_{t_f}$$

which yields the trivial result $\nu = (\lambda_{\lambda_0}^{\lambda_0})_f$. Now, Eq. (50) is equivalent to

$$\left(\frac{\partial \Phi}{\partial t} + \boldsymbol{\lambda}_z^T \dot{\boldsymbol{z}}\right)_{t_f} = 1$$

and because $(\partial\Phi/\partial t)t_f = \nu n_f = (\lambda_{\lambda_0}^{\lambda_0})_f n_f$, and because H^{λ_0} is defined as $H^{\lambda_0} = \boldsymbol{\lambda}_z^T \dot{\boldsymbol{z}}$, we have for the transversality condition at the final time t_f

$$T_r = \left(\lambda_{\lambda_0}^{\lambda_0}\right)_f n_f + H_f^{\lambda_0} = 1 \tag{51}$$

When the set (a, h, k, p, q, λ) is used as in Ref. 6, the augmented Hamiltonian $H^\lambda = -1 + \boldsymbol{\lambda}_z^T \dot{\boldsymbol{z}}$ is not explicitly a function of time such that the system is autonomous and the transversality condition is simply given by $H_f^\lambda = 0$ (Ref. 6), or $H_f^\lambda = 1$ for the unaugmented Hamiltonian $H^\lambda = \boldsymbol{\lambda}_z^T \dot{\boldsymbol{z}}$ used there. However, because

$$\left(\frac{\partial \lambda}{\partial \dot{\boldsymbol{r}}}\right) = \left(\frac{\partial \lambda_0}{\partial \dot{\boldsymbol{r}}}\right) - 3t\dot{\boldsymbol{r}}n^{-1}a^{-2}$$

and $\dot{\lambda} = n + (\partial\lambda/\partial\dot{\boldsymbol{r}})\hat{u} f_t$, and in view of Eq. (38), one recovers $H^\lambda = H^{\lambda_0} + n\lambda_{\lambda_0}^{\lambda_0}$ of Eq. (37) where H^{λ_0} is the Hamiltonian used in this paper. Because $H^\lambda = 1$ throughout the optimal trajectory (Ref. 6), the quantity defined by $n\lambda_{\lambda_0}^{\lambda_0} + H^{\lambda_0} = 1$ remains constant throughout the transfer because it is equivalent to $H^\lambda = 1$. This is especially useful in accepting a converged trajectory as a locally optimal trajectory. Now for the rendezvous problem, the objective function to minimize is taken as follows with the appropriate weights attached to each term if so desired:

$$F' = (a - a_f)^2 + (h - h_f)^2 + (k - k_f)^2 + (p - p_f)^2 + (q - q_f)^2 + (\lambda - \lambda_f)^2 + (T_r - 1)^2$$

We now duplicate the converged minimum-time transfer example of Ref. 16 by running an open-loop trajectory using the present formulation and starting from the low orbit given in Table 4. Because of the relationships given in Eq. (41), the values of the multipliers at time zero must be identical to the values generated in Ref. 16 with the λ formulation. Using $(\lambda_a^{\lambda_0})_0 = 4.675229762$ s/km, $(\lambda_h^{\lambda_0})_0 = 5.413413947 \times 10^2$ s, $(\lambda_k^{\lambda_0})_0 = -9.202702084 \times 10^3$ s, $(\lambda_p^{\lambda_0})_0 = 1.778011878 \times 10$ s, $(\lambda_q^{\lambda_0})_0 = -2.258455855 \times 10^4$ s, $(\lambda_{\lambda_0}^{\lambda_0})_0 = 0$ s/rad and the constant acceleration $f_t = 9.8 \times 10^{-5}$ km/s^2, the equations of motion as well as the adjoint equations are integrated simultaneously using the optimal control $\hat{\boldsymbol{u}}$ from $t = 0$ to t_f = 58,089.90058 s, and with the relative and absolute error controls set to 10^{-9}.

The achieved final conditions are shown in Table 4 with $(H^{\lambda_0})_f = 1.003704076$. These parameters are very close to the target parameters and to the optimized final location $M_f = 46.169264$ deg found in Ref. 16. We now can scale these initial multipliers to get $(H^{\lambda_0})_f = 1$, resulting in the solution shown in Table 4. An open-loop run using these scaled values results in $(H^{\lambda_0})_f = 0.999999393$, which is close

Table 4 Transfer parameters and solution

Orbit	Initial	Target	Achieved	Solution	Initial values
a, km	7,000	42,000	41,999.992	$\lambda_a^{\lambda_0}$, s/km	4.657973438
e	0	10^{-3}	9.983×10^{-4}	$\lambda_h^{\lambda_0}$, s	5.393432977×10^2
i, deg	28.5	1	0.999797	$\lambda_k^{\lambda_0}$, s	-9.168734810×10^3
Ω, deg	0	0	0.000332	$\lambda_p^{\lambda_0}$, s	1.771449217×10
ω, deg	0	0	359.994533	$\lambda_q^{\lambda_0}$, s	-2.250119870×10^4
M_0, M, deg	$M_0 = -130.3331648$ (optimized)	Free	$M = 46.169872$ (optimized)	$\lambda_{\lambda_0}^{\lambda_0}$, s/rad	0.0

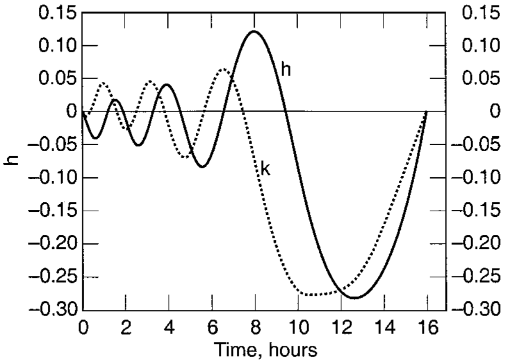

Fig. 2 Time histories of equinoctial elements h and k during minimum-time transfer.

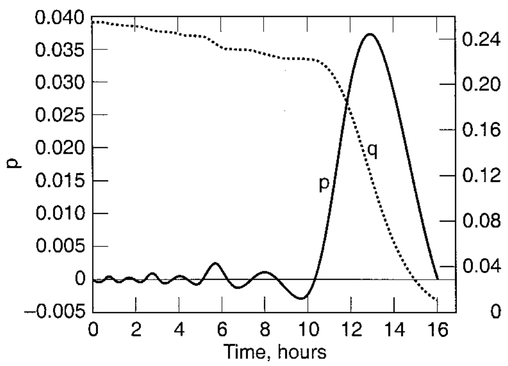

Fig. 3 Time histories of equinoctial elements p and q during minimum-time transfer.

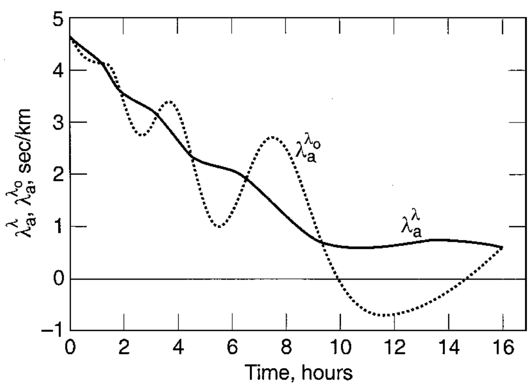

Fig. 4 Evolution of $\lambda_a^{\lambda_0}$ and λ_a^λ multipliers during optimal transfer.

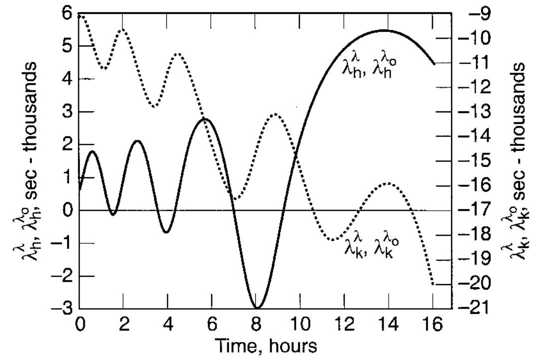

Fig. 5 Evolution of $\lambda_h^{\lambda_0}$, λ_h^λ, and $\lambda_k^{\lambda_0}$, λ_k^λ multipliers during optimal transfer.

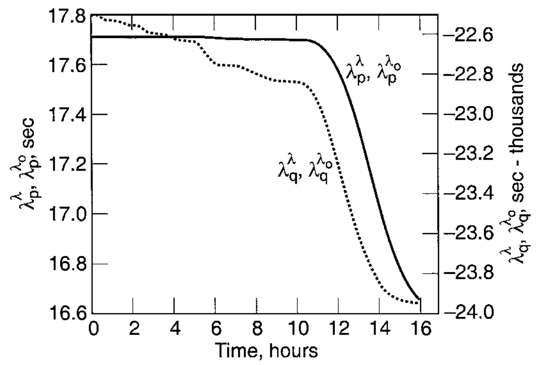

Fig. 6 Evolution of $\lambda_p^{\lambda_0}$, λ_p^λ, and $\lambda_q^{\lambda_0}$, λ_q^λ multipliers during optimal transfer.

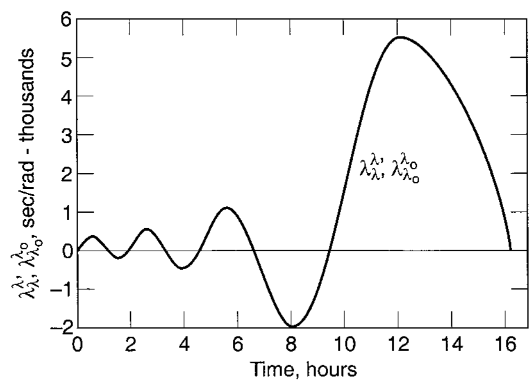

Fig. 7 Evolution of $\lambda_{\lambda_0}^{\lambda_0}$ and λ_λ^λ multipliers during optimal transfer.

to 1. Both Refs. 6 and 16 use the λ formulation, meaning that λ, the mean longitude, is selected as the sixth state variable, the difference being that, unlike Ref. 6, which uses the eccentric longitude F as the accessory variable that defines the radial distance, Ref. 16 uses the true longitude L instead. After correcting an error in the form of h instead of k in Eq. (A-96) of the appendix of Refs. 6, 10, and 11, which should read as in Eq. (A48) of the Appendix in this paper, an open-loop run is carried out using the formulation of Ref. 6, yielding $(\lambda_\lambda^\lambda)_f = 1.63163 \times 10^{-5}$ s/rad, which is close to the value found in Ref. 16 as $(\lambda_\lambda^\lambda)_f = -1.8239 \times 10^{-4}$ s/rad. These values are close to the theoretical value of zero needed for the optimization of the final arrival point on the high orbit. The present λ_0 formulation run yields $(\lambda_{\lambda_0}^{\lambda_0})_f = 8.4844 \times 10^{-3}$ s/rad at the final time, with the quantity $(\lambda_{\lambda_0}^{\lambda_0})n + H^{\lambda_0} = 1.000000015$ constant throughout the transfer and the theoretical transversality condition for the minimum-time transfer $(H^{\lambda_0})_f = 1$ closely matched by $(H^{\lambda_0})_f = 0.999999393$, as mentioned earlier. Figures 2 and 3 show the variations of the elements h, k, p, q on the optimal transfer trajectory whereas Figs. 4–7 depict the evolutions of all six multipliers and compare them with those corresponding to the λ formulation of

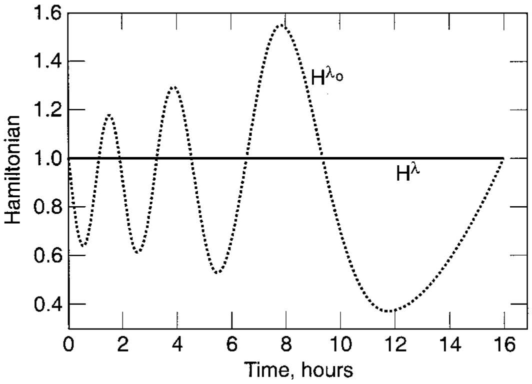

Fig. 8 Variation of H^{λ_0} and constancy of H^λ during optimal transfer.

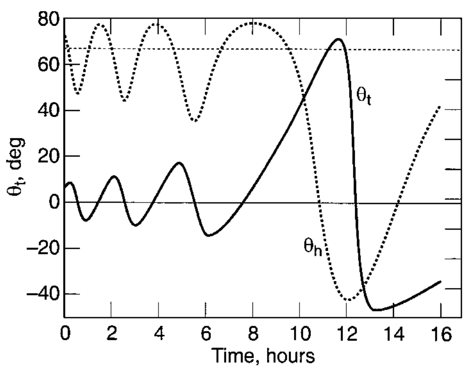

Fig. 9 Optimal-thrust pitch and yaw profiles during minimum-time transfer.

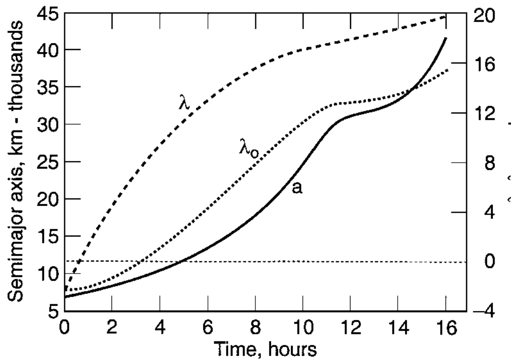

Fig. 10 Semimajor axis change and variation of λ and λ_0 during optimal transfer.

Ref. 6. All of the multipliers have identical values for this transfer example, which requires $(\lambda_{\lambda_0}^{\lambda_0})_0 = (\lambda_\lambda^\lambda)_0 = (\lambda_{\lambda_0}^{\lambda_0})_f = (\lambda_\lambda^\lambda)_f = 0$, and therefore from Eq. (41), $(\lambda_a^{\lambda_0})_0 = (\lambda_a^\lambda)_0$ and $(\lambda_a^{\lambda_0})_f = (\lambda_a^\lambda)_f$ and different λ_a profiles. In Fig. 8, the Hamiltonian of the λ formulation H^λ remains constant throughout, whereas H^{λ_0} is varying but ending at $(H^{\lambda_0})_f = 1$ at the final time. The thrust pitch and yaw profiles are perfectly matched in Fig. 9 with both λ and λ_0 formulations and, finally, the variations of a, λ, and the epoch mean longitude λ_0 are shown in Fig. 10.

Conclusion

The minimum-time rendezvous and transfer problem between given initial and final general elliptic or circular orbits and based on the application of a continuous constant low-thrust acceleration is presented for a particular choice of the fast variable selected as a fundamental epoch element. Besides providing added insight into the mathematics of nonsingular trajectory optimization and mutual validation and verification of the formulations developed thus far, the analysis further reveals that additional simplifications in computing averaged transfers as well as certain aspects of minimum-fuel transfers are possible using this fundamental set. Further comparisons are necessary between the epoch and current time formulations once the specialized software is developed, in order to find out whether certain formulations lead to overall numerically more robust orbital transfer optimization codes. The results presented here are closely related to those published earlier in Ref. 6, which used the current mean longitude as the fast orbital element instead. The mathematical correspondence between the two Hamiltonians and the various multipliers also is established and verified by way of a numerical example. Furthermore, the transformation matrices relating the Lagrange multipliers of the classical elements to the multipliers corresponding to different sets of equinoctial elements also are derived fully.

Appendix: Nonzero Partials of Matrix *M*

Partial Derivatives of *M* with Respect to *h*

The partial derivatives $\partial M_{11}/\partial h$, $\partial M_{12}/\partial h$, $\partial M_{21}/\partial h$, $\partial M_{22}/\partial h$, $\partial M_{23}/\partial h$, $\partial M_{31}/\partial h$, $\partial M_{32}/\partial h$, $\partial M_{33}/\partial h$, $\partial M_{43}/\partial h$, and $\partial M_{53}/\partial h$ are identical to the partials given in Eqs. (A1), (A2), and (A4–A11) of Ref. 11 and therefore they are not repeated here.

$$\frac{\partial M_{61}^0}{\partial h} = n^{-1}a^{-2}\left\{-2\frac{\partial X_1}{\partial h} + 3\frac{\partial \dot{X}_1}{\partial h}t - h\beta G^{-1}\left(h\frac{\partial X_1}{\partial h} + k\frac{\partial X_1}{\partial k}\right) + G\left[\left(\beta + \frac{h^2\beta^3}{1-\beta}\right)\frac{\partial X_1}{\partial h} + \frac{hk\beta^3}{1-\beta}\frac{\partial X_1}{\partial k} + \beta\left(h\frac{\partial^2 X_1}{\partial h^2} + k\frac{\partial^2 X_1}{\partial h\,\partial k}\right)\right]\right\} \tag{A1}$$

$$\frac{\partial M_{62}^0}{\partial h} = n^{-1}a^{-2}\left\{-2\frac{\partial Y_1}{\partial h} + 3\frac{\partial \dot{Y}_1}{\partial h}t - h\beta G^{-1}\left(h\frac{\partial Y_1}{\partial h} + k\frac{\partial Y_1}{\partial k}\right) + G\left[\left(\beta + \frac{h^2\beta^3}{1-\beta}\right)\frac{\partial Y_1}{\partial h} + \frac{hk\beta^3}{1-\beta}\frac{\partial Y_1}{\partial k} + \beta\left(h\frac{\partial^2 Y_1}{\partial h^2} + k\frac{\partial^2 Y_1}{\partial h\,\partial k}\right)\right]\right\} \tag{A2}$$

$$\frac{\partial M_{63}^0}{\partial h} = \frac{G}{na^2}\left[\left(q\frac{\partial Y_1}{\partial h} - p\frac{\partial X_1}{\partial h}\right) + hG^{-2}(qY_1 - pX_1)\right] \tag{A3}$$

Partial Derivatives of *M* with Respect to *k*

The partial derivatives $\partial M_{11}/\partial k$, $\partial M_{12}/\partial k$, $\partial M_{21}/\partial k$, $\partial M_{22}/\partial k$, $\partial M_{23}/\partial k$, $\partial M_{31}/\partial k$, $\partial M_{32}/\partial k$, $\partial M_{33}/\partial k$, $\partial M_{43}/\partial k$, and $\partial M_{53}/\partial k$ are identical to the partials given in Eqs. (A15), (A16), and (A18–A25) of Ref. 11 and they also are not repeated here.

$$\frac{\partial M_{61}^0}{\partial k} = n^{-1}a^{-2}\left\{-2\frac{\partial X_1}{\partial k} + 3\frac{\partial \dot{X}_1}{\partial k}t - k\beta G^{-1}\left(h\frac{\partial X_1}{\partial h} + k\frac{\partial X_1}{\partial k}\right) + G\left[\left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\partial X_1}{\partial k} + \frac{hk\beta^3}{1-\beta}\frac{\partial X_1}{\partial h} + \beta\left(h\frac{\partial^2 X_1}{\partial k\partial h} + k\frac{\partial^2 X_1}{\partial k^2}\right)\right]\right\} \tag{A4}$$

$$\frac{\partial M_{62}^0}{\partial k} = n^{-1}a^{-2}\left\{-2\frac{\partial Y_1}{\partial k} + 3\frac{\partial \dot{Y}_1}{\partial k}t - k\beta G^{-1}\left(h\frac{\partial Y_1}{\partial h} + k\frac{\partial Y_1}{\partial k}\right) + G\left[\left(\beta + \frac{k^2\beta^3}{1-\beta}\right)\frac{\partial Y_1}{\partial k} + \frac{hk\beta^3}{1-\beta}\frac{\partial Y_1}{\partial h} + \beta\left(h\frac{\partial^2 Y_1}{\partial k\partial h} + k\frac{\partial^2 Y_1}{\partial k^2}\right)\right]\right\} \tag{A5}$$

$$\frac{\partial M_{63}^0}{\partial k} = \frac{G^{-1}}{na^2}\left[\left(q\frac{\partial Y_1}{\partial k} - p\frac{\partial X_1}{\partial k}\right) + kG^{-2}(qY_1 - pX_1)\right] \tag{A6}$$

Partial Derivatives of *M* with Respect to *p*

The partial derivatives $\partial M_{23}/\partial p, \partial M_{33}/\partial p, \partial M_{43}/\partial p$, and $\partial M_{53}/\partial p$ are identical to the partials in Eqs. (A29–A32) of Ref. 11, respectively, and

$$\frac{\partial M_{63}^0}{\partial p} = \frac{-X_1}{na^2G} \tag{A7}$$

Partial Derivatives of *M* with Respect to *q*

The partial derivatives $\partial M_{23}/\partial q, \partial M_{33}/\partial q, \partial M_{43}/\partial q$, and $\partial M_{53}/\partial q$ are identical to the partials in Eqs. (A34–A37) of Ref. 11 and

$$\frac{\partial M_{63}^0}{\partial q} = \frac{Y_1}{na^2G} \tag{A8}$$

The partial derivatives of $\dot{X}_1$ with respect to h and k, namely, $\partial\dot{X}_1/\partial h$ and $\partial\dot{X}_1/\partial k$, are identical to those in Eqs. (A39) and (A40) of Ref. 11, and the partials of $\dot{Y}_1$ with respect to h and k, namely, $\partial\dot{Y}_1/\partial h$ and $\partial\dot{Y}_1/\partial k$, are identical to Eqs. (A41) and (A42) of the appendix of Ref. 11. The second partials of X_1 and Y_1 with respect to h and k or $\partial^2X_1/\partial h^2$, $\partial^2X_1/\partial k^2$, $\partial^2X_1/\partial h\partial k$, $\partial^2X_1/\partial k\partial h$, $\partial^2Y_1/\partial h^2$, $\partial^2Y_1/\partial k^2$, $\partial^2Y_1/\partial h\partial k$, and $\partial^2Y_1/\partial k\partial h$ are identical to Eqs. (A43–A50) of Ref. 11. It can be shown that

$$\frac{\partial^2X_1}{\partial h\partial k} = \frac{\partial^2X_1}{\partial k\partial h}, \qquad \frac{\partial^2Y_1}{\partial h\partial k} = \frac{\partial^2Y_1}{\partial k\partial h}$$

Next, the accessory partials $\partial^2X_1/\partial a\partial k$, $\partial^2X_1/\partial a\partial h$, $\partial^2Y_1/\partial a\partial k$, and $\partial^2Y_1/\partial a\partial h$, are

$$\frac{\partial^2X_1}{\partial a\partial k} = \frac{1}{a}\frac{\partial X_1}{\partial k} - \frac{3}{2}\frac{nat}{r}\left[(hs_F + kc_F)\frac{hk\beta^3}{(1-\beta)} + \frac{a^2}{r^2}(k-c_F)(s_F-h\beta) - \frac{a}{r}s_Fc_F\right] \tag{A9}$$

$$\frac{\partial^2X_1}{\partial a\partial h} = \frac{1}{a}\frac{\partial X_1}{\partial h} - \frac{3}{2}\frac{nat}{r}\left\{(hs_F + kc_F)\left[\beta + \frac{h^2\beta^3}{(1-\beta)}\right] - \frac{a^2}{r^2}(h-s_F)(h\beta-s_F) + \frac{a}{r}c_F^2\right] \tag{A10}$$

and

$$\frac{\partial^2Y_1}{\partial a\partial k} = \frac{1}{a}\frac{\partial Y_1}{\partial k} + \frac{3}{2}\frac{nat}{r}\left\{(hs_F + kc_F)\left[\beta + \frac{k^2\beta^3}{(1-\beta)}\right] + \frac{a^2}{r^2}(k-c_F)(c_F-k\beta) + \frac{a}{r}s_F^2\right\} \tag{A11}$$

$$\frac{\partial^2Y_1}{\partial a\partial h} = \frac{1}{a}\frac{\partial Y_1}{\partial h} + \frac{3}{2}\frac{nat}{r}\left[(hs_F + kc_F)\frac{hk\beta^3}{(1-\beta)} - \frac{a^2}{r^2}(h-s_F)(k\beta-c_F) - \frac{a}{r}s_Fc_F\right] \tag{A12}$$

Partial Derivatives of *M* with Respect to *a*

The partial derivatives are written in terms of the first-order partials of $X_1, \dot{X}_1$ and $Y_1, \dot{Y}_1$ with respect to a as well as $\partial^2X_1/\partial a\partial h$, $\partial^2X_1/\partial a\partial k$, $\partial^2Y_1/\partial a\partial h$, and $\partial^2Y_1/\partial a\partial k$ shown below and in Eqs. (A9–A12).

$$\frac{\partial M_{11}}{\partial a} = \frac{4}{n^2a^2}\dot{X}_1 + \frac{2}{n^2a}\frac{\partial\dot{X}_1}{\partial a} \tag{A13}$$

$$\frac{\partial M_{12}}{\partial a} = \frac{4}{n^2a^2}\dot{Y}_1 + \frac{2}{n^2a}\frac{\partial\dot{Y}_1}{\partial a} \tag{A14}$$

$$\frac{\partial M_{21}}{\partial a} = \frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial X_1}{\partial k} + \frac{\partial^2X_1}{\partial a\partial k} - \frac{h\beta}{na}\dot{X}_1 - \frac{h\beta}{n}\frac{\partial\dot{X}_1}{\partial a}\right] \tag{A15}$$

$$\frac{\partial M_{22}}{\partial a} = \frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial Y_1}{\partial k} + \frac{\partial^2Y_1}{\partial a\partial k} - \frac{h\beta}{na}\dot{Y}_1 - \frac{h\beta}{n}\frac{\partial\dot{Y}_1}{\partial a}\right] \tag{A16}$$

$$\frac{\partial M_{23}}{\partial a} = \frac{k}{na^2G}\left[-\frac{1}{2a}(qY_1 - pX_1) + q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right] \tag{A17}$$

$$\frac{\partial M_{31}}{\partial a} = -\frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial X_1}{\partial h} + \frac{\partial^2X_1}{\partial a\partial h} + \frac{k\beta}{na}\dot{X}_1 + \frac{k\beta}{n}\frac{\partial\dot{X}_1}{\partial a}\right] \tag{A18}$$

$$\frac{\partial M_{32}}{\partial a} = -\frac{G}{na^2}\left[-\frac{1}{2a}\frac{\partial Y_1}{\partial h} + \frac{\partial^2Y_1}{\partial a\partial h} + \frac{k\beta}{na}\dot{Y}_1 + \frac{k\beta}{n}\frac{\partial\dot{Y}_1}{\partial a}\right] \tag{A19}$$

$$\frac{\partial M_{33}}{\partial a} = \frac{-h}{na^2G}\left[-\frac{1}{2a}(qY_1 - pX_1) + q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right] \tag{A20}$$

$$\frac{\partial M_{43}}{\partial a} = \frac{K}{2na^2G}\left(-\frac{1}{2a}Y_1 + \frac{\partial Y_1}{\partial a}\right) \tag{A21}$$

$$\frac{\partial M_{53}}{\partial a} = \frac{K}{2na^2G}\left(-\frac{1}{2a}X_1 + \frac{\partial X_1}{\partial a}\right) \tag{A22}$$

$$\frac{\partial M_{61}^0}{\partial a} = -\frac{M_{61}^0}{2a} + n^{-1}a^{-2}\left[-2\frac{\partial X_1}{\partial a} + 3\frac{\partial\dot{X}_1}{\partial a}t + G\left(h\beta\frac{\partial^2X_1}{\partial a\partial h} + k\beta\frac{\partial^2X_1}{\partial a\partial k}\right)\right] \tag{A23}$$

$$\frac{\partial M_{62}^0}{\partial a} = -\frac{M_{62}^0}{2a} + n^{-1}a^{-2}\left[-2\frac{\partial Y_1}{\partial a} + 3\frac{\partial\dot{Y}_1}{\partial a}t + G\left(h\beta\frac{\partial^2Y_1}{\partial a\partial h} + k\beta\frac{\partial^2Y_1}{\partial a\partial k}\right)\right] \tag{A24}$$

$$\frac{\partial M_{63}^0}{\partial a} = -\frac{M_{63}^0}{2a} + \frac{1}{na^2}\left[\left(q\frac{\partial Y_1}{\partial a} - p\frac{\partial X_1}{\partial a}\right)(1-h^2-k^2)^{-\frac{1}{2}}\right] \tag{A25}$$

with

$$\frac{\partial\dot{X}_1}{\partial a} = -\frac{1}{2}\frac{na}{r}\left[1 - \frac{3na^2t}{r^2}(ks_F - hc_F)\right]\left[hk\beta c_F - (1-h^2\beta)s_F\right] + \frac{3}{2}\frac{n^2a^2t}{r^2}\left[hk\beta s_F + (1-h^2\beta)c_F\right] \tag{A26}$$

$$\frac{\partial\dot{Y}_1}{\partial a} = \frac{1}{2}\frac{na}{r}\left[1 - \frac{3na^2t}{r^2}(ks_F - hc_F)\right]\left[hk\beta s_F - (1-k^2\beta)c_F\right] + \frac{3}{2}\frac{n^2a^2t}{r^2}\left[hk\beta c_F + (1-k^2\beta)s_F\right] \tag{A27}$$

$$\frac{\partial X_1}{\partial a} = \frac{X_1}{a} - \frac{3}{2}\frac{t}{a}\dot{X}_1 \tag{A28}$$

$$\frac{\partial Y_1}{\partial a} = \frac{Y_1}{a} - \frac{3}{2}\frac{t}{a}\dot{Y}_1 \tag{A29}$$

Partial Derivatives of *M* with respect to λ_0

$$\frac{\partial M_{11}}{\partial\lambda_0} = \frac{\partial M_{11}}{\partial\lambda}\frac{\partial\lambda}{\partial\lambda_0} = \frac{\partial M_{11}}{\partial\lambda} = \frac{2}{n^2r}\frac{\partial\dot{X}_1}{\partial F} \tag{A30}$$

$$\frac{\partial M_{12}}{\partial\lambda_0} = \frac{2}{n^2r}\frac{\partial\dot{Y}_1}{\partial F} \tag{A31}$$

$$\frac{\partial M_{21}}{\partial \lambda_0} = \frac{G}{nar}\left(\frac{\partial^2 X_1}{\partial F \partial k} - \frac{h\beta}{n}\frac{\partial \dot{X}_1}{\partial F}\right) \tag{A32}$$

$$\frac{\partial M_{22}}{\partial \lambda_0} = \frac{G}{nar}\left(\frac{\partial^2 Y_1}{\partial F \partial k} - \frac{h\beta}{n}\frac{\partial \dot{Y}_1}{\partial F}\right) \tag{A33}$$

$$\frac{\partial M_{23}}{\partial \lambda_0} = k\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)\bigg/ narG \tag{A34}$$

$$\frac{\partial M_{31}}{\partial \lambda_0} = -\frac{G}{nar}\left(\frac{\partial^2 X_1}{\partial F \partial h} + \frac{k\beta}{n}\frac{\partial \dot{X}_1}{\partial F}\right) \tag{A35}$$

$$\frac{\partial M_{32}}{\partial \lambda_0} = -\frac{G}{nar}\left(\frac{\partial^2 Y_1}{\partial F \partial h} + \frac{k\beta}{n}\frac{\partial \dot{Y}_1}{\partial F}\right) \tag{A36}$$

$$\frac{\partial M_{33}}{\partial \lambda_0} = -h\left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)\bigg/ narG \tag{A37}$$

$$\frac{\partial M_{43}}{\partial \lambda_0} = \frac{K}{2narG}\frac{\partial Y_1}{\partial F} \tag{A38}$$

$$\frac{\partial M_{53}}{\partial \lambda_0} = \frac{K}{2narG}\frac{\partial X_1}{\partial F} \tag{A39}$$

$$\frac{\partial M_{61}^0}{\partial \lambda_0} = n^{-1}a^{-1}r^{-1}\left[-2\frac{\partial X_1}{\partial F} + 3\frac{\partial \dot{X}_1}{\partial F}t + G\left(h\beta\frac{\partial^2 X_1}{\partial F \partial h} + k\beta\frac{\partial^2 X_1}{\partial F \partial k}\right)\right] \tag{A40}$$

$$\frac{\partial M_{62}^0}{\partial \lambda_0} = n^{-1}a^{-1}r^{-1}\left[-2\frac{\partial Y_1}{\partial F} + 3\frac{\partial \dot{Y}_1}{\partial F}t + G\left(h\beta\frac{\partial^2 Y_1}{\partial F \partial h} + k\beta\frac{\partial^2 Y_1}{\partial F \partial k}\right)\right] \tag{A41}$$

$$\frac{\partial M_{63}^0}{\partial \lambda_0} = \left(q\frac{\partial Y_1}{\partial F} - p\frac{\partial X_1}{\partial F}\right)\bigg/ narG \tag{A42}$$

The auxiliary partials are

$$\frac{\partial X_1}{\partial F} = a\left[hk\beta c_F - (1-h^2\beta)s_F\right] \tag{A43}$$

$$\frac{\partial Y_1}{\partial F} = a\left[-hk\beta s_F + (1-k^2\beta)c_F\right] \tag{A44}$$

$$\frac{\partial \dot{X}_1}{\partial F} = -\frac{a}{r}(ks_F - hc_F)\dot{X}_1 + \frac{a^2 n}{r}\left[-hk\beta s_F - (1-h^2\beta)c_F\right] \tag{A45}$$

$$\frac{\partial \dot{Y}_1}{\partial F} = -\frac{a}{r}(ks_F - hc_F)\dot{Y}_1 + \frac{a^2 n}{r}\left[-hk\beta c_F - (1-k^2\beta)s_F\right] \tag{A46}$$

$$\frac{\partial^2 X_1}{\partial F \partial h} = a\left[(hs_F + kc_F)\left(\beta + \frac{h^2\beta^3}{1-\beta}\right) + \frac{a^2}{r^2}(h\beta - s_F)(s_F - h) + \frac{a}{r}c_F^2\right] \tag{A47}$$

$$\frac{\partial^2 X_1}{\partial F \partial k} = -a\left[-(hs_F + kc_F)\frac{hk\beta^3}{1-\beta} + \frac{a^2}{r^2}(s_F - h\beta)(c_F - k) + \frac{a}{r}s_F c_F\right] \tag{A48}$$

$$\frac{\partial^2 Y_1}{\partial F \partial h} = a\left[-(hs_F + kc_F)\frac{hk\beta^3}{1-\beta} - \frac{a^2}{r^2}(k\beta - c_F)(s_F - h) + \frac{a}{r}s_F c_F\right] \tag{A49}$$

$$\frac{\partial^2 Y_1}{\partial F \partial k} = a\left[-(hs_F + kc_F)\left(\beta + \frac{k^2\beta^3}{1-\beta}\right) + \frac{a^2}{r^2}(c_F - k\beta)(c_F - k) - \frac{a}{r}s_F^2\right] \tag{A50}$$

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